Cauchy’s Theorem for Abelian Groups
If $G$ is a finite abelian group and $p$ is a prime that divides $|G|$, then $\exists g \in G$ such that $|g| = p$.

Proof.

We shall use Strong Induction on the order of $G$ to prove it. When $|G| = 2$, the only prime that divides $|G|$ is 2. Let $g$ be a nonidentity element in $G$, then $g^2$ is the identity, hence $|g| = 2$. Now assume the theorem holds for all abelian groups of order less than $n$ and suppose $|G| = n$. Let $a$ be any nonidentity element of $G$. Then the order of $a$ is a positive integer and is therefore divisible by some prime $q$ (by the Fundamental Theorem of Arithmetic). Then $|a| = qt$ for some positive integer $t$. Let $b = a^t$, then $|b| = q$. If $q = p$, then we are done.

If $q \neq p$, let $N$ be cyclic subgroup $\langle b \rangle$. Since $G$ is abelian, $N$ is normal and $|N| = q$. Then $|G/N| = |G|/|N| = n/q$ (by Lagrange’s Theorem). But $n/q < n$. Thus, by the induction hypothesis, the theorem is true for $G/N$. Note that $|G| = |N||G/N| = q|G/N|$. Since $q \neq p$, $p$ divides $|G/N|$. Thus, $G/N$ contains an element of order $p$, say, $Nc$. Note that $Nc^p = (Nc)^p = Ne$ (where $e$ denotes the identity of $G$), thus, $c^p \in N$. Also,
\[ c^{pq} = (c^p)^q = e. \] Thus, \( c \) must have order dividing \( pq \). Note that \( c \) cannot have order 1, for otherwise \( Nc \) would have order 1 instead of \( p \). Also, \( c \) cannot have order \( q \), for otherwise \( (Nc)^q = N \implies p|q \), contradicting the fact that \( q \) is a prime different from \( p \). Thus, we are left with the possibilities that \( |c| = p \) or \( |c| = pq \). In the first case, set \( g = c \). If it is the latter case, set \( g = c^q \). Therefore, the theorem holds for abelian groups of order \( n \), for any positive integer \( n \).

Q.E.D.